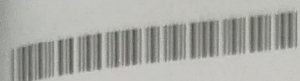


K22P 1410



Reg. No. : .....

Name : .....

III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.)

Examination, October 2022

(2019 Admission Onwards)

MATHEMATICS

MAT 3C13 : Complex Function Theory

Max. Marks : 80

Time : 3 Hours

PART – A

Attempt **any four** questions from this Part. **Each** question carries **4** marks.

1. Define the following terms :

- i) Period module of a meromorphic function
- ii) Discrete module.

2. Show that the series  $\sum_{n=1}^{\infty} n^{-z}$  converges uniformly and absolutely on a subset of the complex plane  $\mathbb{C}$ .

3. Is  $\mathbb{C} - \{0\}$  simply connected ? Justify your answer.

4. Is the sets  $\{z : |z| < 1\}$  and  $\mathbb{C}$  homeomorphic ? Justify your answer.

5. Prove that a harmonic function  $u$  in  $\mathbb{C}$  is infinitely differentiable.

6. Given that  $v_1$  and  $v_2$  are two harmonic conjugates of a harmonic function  $u$ .  
Prove that  $v_2 - v_1 = c$ , where  $c$  is a constant.

P.T.O.

## PART - B

Answer any four questions from this Part without omitting any Unit. Each question carries 16 marks.

## Unit - I

7. a) Prove the following :

i) Let  $S = \{z : \operatorname{Re} z \geq a\}$  where  $a > 1$ . If  $\varepsilon > 0$ , then there is a number  $\delta > 0$ ,  $0 < \delta < 1$ , such that for all  $z \in S$ ,  $\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon$  whenever  $\delta > \beta > \alpha$ .

ii) Let  $S = \{z : \operatorname{Re} z \leq A\}$  where  $-\infty < A < \infty$ . If  $\varepsilon > 0$ , then there is a number  $k > 1$  such that for all  $z \in S$ ,  $\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon$  whenever  $\beta > \alpha > k$ .

b) Prove : A non-constant elliptic function has equally many poles as it has zeroes.

8. With the usual notations, prove that :

a)  $\wp(2z) = \frac{1}{4} \left( \frac{\wp''(z)}{\wp'(z)} \right)^2 - 2\wp(z)$

b)  $\wp'(z) = -\sigma(2z) / \sigma(z)^4$

c) 
$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0$$

d)  $\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$

9. a) Prove that Riemann's zeta function  $\zeta$  has no other zeroes outside the closed strip  $\{z : 0 \leq \operatorname{Re} z \leq 1\}$ .

b) Prove that if  $\operatorname{Re} z > 1$ , then  $\zeta(z) = \prod_{n=1}^{\infty} \left( \frac{1}{1 - p_n^{-z}} \right)$  where  $p_n$  is a sequence of prime numbers.



### Unit – II

10. State and prove Schwarz Reflection Principle.
11. a) Let  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be a path and let  $\{(f_t, D_t) : 0 \leq t \leq 1\}$  be an analytic continuation along  $\gamma$ . Show that  $\{(f'_t, D_t) : 0 \leq t \leq 1\}$  is also a continuation along  $\gamma$ .
- b) Let  $(f, D)$  be a function element which admits unrestricted continuation in the simply connected region  $G$ . Prove that there is an analytic function  $F : G \rightarrow \mathbb{C}$  such that  $F(z) = f(z)$  for all  $z$  in  $D$ .
- c) Is the region  $\{z \in \mathbb{C} : 1 < |z| < 2\}$  simply connected? Justify your answer.
12. State and prove the Mittag-Leffler's theorem.

### Unit – III

13. a) State and prove Jensen's formula.
- b) State and prove Maximum Principle (Second Version).
14. Prove that the Dirichlet problem can be solved in a unit disk.
15. a) Define the Poisson kernel  $P_r(\theta)$ . Prove that  $P_r(\theta) = \operatorname{Re} \left( \frac{1+re^{i\theta}}{1-re^{i\theta}} \right)$ .
- b) Prove that  $P_r(\theta) < P_r(\delta)$  if  $0 < \delta < |\theta| \leq \pi$ .
- c) For  $|z| < 1$  let  $u(z) = \operatorname{Im} \left[ \left( \frac{1+z}{1-z} \right)^2 \right]$ . Show that  $u$  is harmonic.